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CONTROL IN MIXED STRATEGIES ON THE MINIMAX OF AN INTEGRAL FUNCTIONAL†

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The problem of feedback-implementable optimal control, on condition that an integral quality index takes a minimax value [1–7], is solved by mixed strategies. An effective procedure is proposed for evaluating the value of the game and for constructing optimal strategies, based on the idea of stochastic programmed synthesis [1, 7]. An essential element of the procedure is the reduction of multidimensional auxiliary problems to the maximization or minimization of functions whose arguments are of dimensions not exceeding that of the controlled system.

1. STATEMENT OF THE PROBLEM

CONSIDER an object described by the differential equation

$$x^{\cdot}=A(t)x+f(t, u, v), \quad t_0\leqslant t\leqslant\vartheta$$
$$x\in R^n, \quad u\in W_u\subset R^r, \quad v\in W_v\subset R^s \tag{1.1}$$

where W_u and W_v are compact sets, $A(t)$ is a matrix-valued function and $f(t, u, v)$ is a vector-valued function, both piecewise continuous in t. Times $t_*^{[i]}$, seminorms $\mu_*^{[i]}(x)$ $(i = 1, \ldots, N_*)$, $t^{[N_*]} = \vartheta$ and a seminorm-function $\mu_*(t, x)$, piecewise continuous in t, are defined over the interval $[t_0, \vartheta]$. (Such functions are assumed to be right continuous.) Suppose a set of integers $\nu^{[i]} = \nu[t_*^{[i]}]\in[1, n]$ and constant $(\nu^{[i]}\times n)$-matrices $D_*^{[i]}$ are given. The seminorm $\mu_*^{[i]}(x[t_*^{[i]}])$ is defined as a certain norm $\mu_*^{[i]}(D_*^{[i]}x[t_*^{[i]}])$, $(i = 1, \ldots, N_*)$. Similarly, $\mu_*(t,x)$ is defined as a norm-function $\mu(t, D_*(t)x)$ which is piecewise continuous in t, where $D_*(t)$ is a piecewise constant matrix-valued function of order $(\nu[t]\times n)$, $\nu[t]\in[1, n]$, $t_0\leqslant t\leqslant\vartheta$.

The problem is to determine controls u and v which, respectively, minimize and maximize a quality index

$$\gamma = \int_{t_*}^{\vartheta}\mu(t, D_*(t)x[t])\,dt + \sum_{i=g_*}^{N_*}\mu^{[i]}(D_*^{[i]}x[t_*^{[i]}]), \quad t_*\in[t_0, \vartheta) \tag{1.2}$$

where t_* is the starting time of the control process and $t_*^{[g_*]}$ is the least of the time $t_*^{[i]}\geqslant t_*$. The problem will be solved in the class of mixed positional strategies [7]

$$S^u=\{U_v(\cdot),\ p_v(\cdot);\ U_v^*(\cdot),\ p_v^*(\cdot);\ V_v^*(\cdot),\ q_v^*(\cdot)\} \tag{1.3}$$

$$S^v=\{V_z(\cdot),\ q_z(\cdot);\ U_z^*(\cdot),\ p_z^*(\cdot);\ V_z^*(\cdot),\ q_z^*(\cdot)\} \tag{1.4}$$

$$U(\cdot)=\{U(\varepsilon)=\{u^{[r]}\in W_u,\ r=1, \ldots, L_\varepsilon\},\ \varepsilon>0\}$$

$$V(\cdot)=\{V(\varepsilon)=\{v^{[s]}\in W_v,\ s=1, \ldots, M_\varepsilon\},\ \varepsilon>0\}$$

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$$p_y(\cdot)=\left\{p_r=p_r(t,x,y,\varepsilon)\geqslant 0,\quad \sum_{r=1}^{L_\varepsilon}p_r=1\right\}$$

$$q_z(\cdot)=\left\{q_s=q_s(t,x,z,\varepsilon)\geqslant 0,\quad \sum_{s=1}^{M_\varepsilon}q_s=1\right\}$$

($\varepsilon>0$ is an accuracy parameter [1]). The control scheme is such that, together with the x-object (1.1), one also considers y, a pilot model installed in the control organ R_u, and z, a pilot model installed in the control system R_v. Underlying the construction of the motions of the $x[\cdot]$-object and the $y[\cdot]$-model is a certain probability space $\{\Omega_*, F_*, P_*\}$ which is constructed on the basis of the functions $p_y(\cdot)$, $q_z(\cdot)$ in (1.3) and the properties of random interference

$$v[\cdot]=\{v[t,\omega]\in W_v,\ t_*\leqslant t\leqslant\vartheta,\ \omega\in\Omega_*\} \tag{1.5}$$

For an initial position $\{t_*, x_*, y_*\}$, a given value of $\varepsilon>0$, a partition $\Delta\{t_i\}=\{t_1=t_*,\ t_i<t_{i+1},\ t_{k+1}=\vartheta\}$ and a certain interference $v[\cdot]$ as in (1.5), the strategy S^u of (1.3) will generate motions $x[\cdot]$ and $y[\cdot]$ as solutions of stepwise differential equations

$$x^{\cdot}[t,\omega]=A(t)x[t,\omega]+f(t,u[t_i,\omega],v[t,\omega])$$
$$t_i<t<t_{i+1},\quad i=1,\dots,k,\quad x[t_1,\omega]=x_* \tag{1.6}$$

$$y^{\cdot}[t,\omega]=A(t)y[t,\omega]+\sum_{r,s=1}^{L_\varepsilon,M_\varepsilon}f(t,u^{[r]},v^{[s]})\,p_r^*(t_i,x[t_i,\omega],\ y[t_i,\omega],\varepsilon)\times$$
$$\times\, q_s^*(t_i,x[t_i,\omega],\ y[t_i,\omega],\varepsilon) \tag{1.7}$$

The function $u[t_1,\omega]$ in (1.6) is a sample function of the random variable $u[t_1,\cdot]$ such that

$$P(u[t_i,\omega]=u^{[r]}|x[t_i,\omega],y[t_i,\omega])=$$
$$=p_r(t_i,x[t_i,\omega],y[t_i,\omega],\varepsilon),\quad u^{[r]}\in U(\varepsilon)$$

where $P(\cdots|\cdots)$ is the conditional probability. It is assumed that the noise is stochastically independent of the control at each step, i.e.,

$$P(v[t,\omega]\in C|x[t_i,\omega],\ y[t_i,\omega],u[t_i,\omega])=$$
$$=\mathrm{Idem}(u[t_i,\omega])\to))$$

Throughout this paper "Idem" on the right of an equality will stand for an expression that is identical with the left-hand side except for the substitution of symbols indicated in the parentheses.

The guaranteed result is defined to be

$$\rho(S^u,t_*,x_*)=\lim_{\beta\to1}\overline{\lim_{\varepsilon\to0}}\lim_{\xi\to0}\sup_{|x_*-y_*|\leqslant\xi}\lim_{\delta\to0}\sup_{\Delta_\delta}\sup_{v[\cdot]}(\min\alpha) \tag{1.8}$$

where α satisfies the condition

$$P(\gamma(x[\cdot])\leqslant\alpha)\geqslant\beta$$

A strategy $S_0{}^u$ is optimal if

$$\rho(S^u,t_*,x_*)=\min_{S^u}\rho(S^u,t_*,x_*)=\rho_u{}^0(t_*,x_*) \tag{1.9}$$

for any position $\{t_*, x_*\}$. The quantity $\rho_u{}^\circ(\cdot)$ is the optimal guaranteed result. Similarly one defines $\rho(S^v, t_*, x_*)$ and optimal $S_0{}^v$ and $\rho_v{}^\circ(t_*, x_*)$:

$$\rho(S_0{}^v,t_*,x_*)=\max_{S^v}\rho(S^v,t_*,x_*)=\rho_v{}^\circ(t_*,x_*)$$

The quality index γ of (1.2) is positional, and it can be shown that the differential game for the object (1.1) with quality index (1.2) has a payoff $\rho^\circ(\cdot) = \rho_u^\circ(\cdot) = \rho_v^\circ(\cdot)$ and saddle point $\{S_0^u, S_0^v\}$.

The optimal strategies S_0^u (1.3), (1.9) and S_0^v (1.4), (1.10) are built up constructively on the basis of the known payoff function $\rho^\circ(t, x)$ of the game by the method of extremal shift to satellite points [6]. The purpose of this paper is to describe an effective construction for the payoff of the game and the optimal strategies, based on the idea of stochastic programmed synthesis [1, 7].

2. STOCHASTIC PROGRAMMED MAXIMIN

We denote a partition $\Delta\{\tau_*^{[h]}\}$ of the interval $[t_0, \vartheta]$: $\tau_*^{[1]} = t_0$, $\tau_*^{[h+1]} > \tau_*^{[h]}$, $\tau_*^{[M]} = \vartheta$, also including in it all the points t_j° that separate the intervals in which $D_*(t)$ is constant. Define

$$\mu(\tau_*^{[h]}, D(\tau^{[h-1]})x) = \int\limits_{\tau_*^{[h-1]}}^{\tau_*^{[h]}} \mu(\tau, D_*(\tau)x)\,d\tau$$

We will dwell only on the values of $\tau_*^{[h]}$ at which $\mu(\tau_*^{[h]}, D(\tau_*^{[h-1]})x) \neq 0$. Combining the times $t_*^{[i]}$, $\tau_*^{[h]}$ we renumber them in increasing order and denote them by $t^{[i]}$, $i = 1, \ldots, N$. Define

$$\mu^{[i]}(D^{[i]}x) = \begin{cases} \mu(\tau_*^{[h]}, & D(\tau_*^{[h-1]})x), \quad t^{[i]} = \tau_*^{[h]} \\ \mu^{[j]}(D_*^{[j]}x), & t^{[i]} = t_*^{[j]} \end{cases}$$

To simplify the notation we shall assume that the combined times are all distinct. Otherwise certain obvious changes must be made in the constructions described below, due to the double allowance for μ at times $t^{[i]} = \tau_*^{[h]} = t_*^{[j]}$.

Let w be the model described by the equation

$$w^\cdot = A(t)w + \sum_{r,s=1}^{L_\eta, M_\eta} f(t, u^{[r]}, v^{[s]})p_r q_s \tag{2.1}$$

where

$$p_r \geqslant 0, \sum_{r=1}^{L_\eta} p_r = 1, \quad q_s \geqslant 0, \sum_{s=1}^{M_\eta} q_s = 1 \tag{2.2}$$

$$u^{[r]} \in U_\eta = \{u^{[r]} \in W_u, \ r = 1, \ldots, L_\eta\}$$

$$v^{[s]} \in V_\eta = \{v^{[s]} \in W_v, \ s = 1, \ldots, M_\eta\}$$

The set U_η has the property that for any $u \in W_u$ there exists $u^{[r]} \in U_\eta$, $|u^{[r]} - u| \leqslant \eta$. Similarly, V_η is a set such that for any $v \in W_v$ there exists $v^{[s]} \in V_\eta$, $|v^{[s]} - v| \leqslant \eta$. Let $\{\tau_*, w_*\}$ be the starting position for the stochastic model (2.1).

We denote another partition

$$\Delta = \Delta\{\tau_j\} = \{\tau_1 = \tau_*, \ \tau_j < \tau_{j+1}, \ \tau_{k+1} = \vartheta\} \tag{2.3}$$

which, besides the previous points, also includes all times $t^{[i]} \geqslant \tau_*$. Let $t^{[g]}$ be the least of the times $t^{[i]} \geqslant \tau_*$. The model (2.1) is based on a probability space [8] $\{\Omega, B, P\}$ in which the elementary events are $\omega = \{\xi_1, \ldots, \xi_k\}$, where $\xi_j \in [0, 1]$ are uniformly distributed independent random variables associated with τ_j. Here $\Omega = \{\omega\}$ is the unit cube in k-space, B is a Borel σ-algebra for the cube and $P = P(B_*)$ is a Lebesque measure, $B_* \in B$. We introduce $\nu^{[i]}$-dimensional vector-valued random variables $l^{(i)}(t^{[i]}, \omega)$ defined on $\{\Omega, B, P\}$, where $\nu^{[i]} \in [1, n]$ is defined by the order of the matrix $D^{[i]}$ corresponding to the time $t^{[i]}$, so that $\nu^{[i]}$ is the number of rows of $D^{[i]}$. Combining all the random variables $l^{(i)}(\cdot)$, we get a multidimensional random variable $l(\cdot)$. Define

$$\|l(\cdot)\| = \max_i \operatorname{vrai}\max_\omega \mu^{[i]*}(l^{(i)}(t^{[i]}, \omega))$$

where $\mu^{[i]*}(l)$ is the norm dual to $\mu^{[i]}(l)$. Let $X(t, \tau)$ denote a fundamental matrix for $dx/dt = A(t)x$, $M\{\cdots\}$ and $M\{\cdots|\cdots\}$ the expectation and conditional expectation, respectively and $\langle a \cdot b\rangle = a^T b$ the scalar product of the vectors a and b. Throughout what follows, lower-case Latin letters will denote column vectors and the superscript T transposition, so that a^T is a row vector.

Our main result is the following procedure and a proof of its validity. It can be verified that

$$\rho^\circ(\tau_*, w_*) = \lim_{\eta \to 0,\ \delta \to 0} e(\tau_*, w_*, \Delta, \eta) \tag{2.4}$$

$$\delta = \max_{j,h} [(\tau_{j+1} - \tau_j), \quad (\tau_*^{[h+1]} - \tau_*^{[h]})]$$

where η is a number defining sets U_η and V_η of vectors $u^{[r]}$ and $v^{[s]}$ distributed fairly densely in the compact sets W_u and W_v and $\rho^\circ(\cdot)$ is the payoff of the game (1.9), (1.10). The quantity $e(\cdot)$ is defined by

$$e(\tau_*, w_*, \Delta, \eta) = \max_{\|l(\cdot)\| \leq 1} \Big[\sum_{i=g}^{N} M\{l^{(i)T}(t^{[i]}, \omega)\} D^{[i]} X(t^{[i]}, \tau_*) w_* +$$
$$+ M\Big\{ \sum_{j=1}^{k} \int_{\tau_j}^{\tau_{j+1}} \min_{u \in W_u} \max_{v \in W_v} \Big[M\Big\{ \sum_{i=d(j)}^{N} l^{(i)T}(t^{[i]}, \omega) D^{[i]} X(t^{[i]}, \tau) \times$$
$$\times \sum_{r,s=1}^{L_\eta, M_\eta} f(\tau, u^{[r]}, v^{[s]}) p_r q_s \,|\, \xi_1, \ldots, \xi_j \Big\} \Big] d\tau \Big\} \Big] \tag{2.5}$$

where $d(j) = \min(i)$ such that $t^{[i]} \geq \tau_{j+1}$, $h = g$ if $\tau_* < t^{[g]}$, otherwise $h = g + 1$.

Putting

$$w^* = X(\vartheta, \tau_*) w_*, \quad f^*(\tau, u, v) = X(\vartheta, \tau) f(\tau, u, v)$$

$$m^{(i)} = X^T(t^{[i]}, \vartheta) M\{D^{[i]T} l^{(i)}\}, \quad \mu^{[i]*}(l^{(i)}) \leq 1$$

$$m_j^{(i)T} = M\{l^{(i)T}(t^{[i]}, \omega) D^{[i]} X(t^{[i]}, \vartheta) \,|\, \xi_1, \ldots, \xi_j\}$$

we obtain

$$e(\tau_*, w_*, \Delta, \eta) = \max_{m^*} [m^{*T} w_* + \varkappa(\tau_*, m_*, \Delta, \eta)] \tag{2.6}$$

$$\varkappa(\tau_*, m_*, \Delta, \eta) = \max_{\|l(\cdot)\| \leq 1} M \Big\{ \sum_{j=1}^{k} \int_{\tau_j}^{\tau_{j+1}} \min_p \max_q \Big(\sum_{i=d(j)}^{N} m^{(i)T} \Big) \times$$
$$\times \Big(\sum_{r,s=1}^{L_\eta, M_\eta} f^*(\tau, u^{[r]}, v^{[s]}) p_r q_s \Big) d\tau \Big\}, \quad m_* = \sum_{i=h}^{N} m^{(i)} \tag{2.7}$$

where m_* is a constituent of m^*; the minimax is evaluated over all arguments satisfying conditions (2.2).

To evaluate $\varkappa(\cdot)$, one can develop and justify a procedure [7] that uses induction on j to construct the upward convex envelopes $\varphi_j^{(i)}(m)$ for certain functions $\psi_j(m)$ of the appropriate conditional expectations m. We define

$$\varphi_{k+1}^{(N)}(m) = 0, \quad \Delta\psi_k(m) = I(\tau_k, \tau_{k+1}, m) =$$
$$= \int_{\tau_k}^{\tau_{k+1}} \min_p \max_q \Big[m^T \sum_{r,s=1}^{L_\eta, M_\eta} f^*(\tau, u^{[r]}, v^{[s]}) p_r q_s \Big] d\tau$$

$$\psi_k(m) = \Delta\psi_k(m)$$

If we define $m_N = D^{[N]T}l^{(N)}(t^{[N]}, \omega)$, then, using the total probability formula, we see that for each conditional expectation

$$m_k = M\{D^{[N]T}l^{(N)}(t^{[N]}, \omega) \mid \xi_1, \ldots, \xi_{k-1}\}$$

we must maximize $M\{\Delta\psi_k(m_N) \mid \xi_1, \ldots, \xi_{k-1}\}$. This maximum value is by definition the value of the upward convex envelope $\varphi_k(m_k)$ of $\psi_k(\cdot)$ at the point m_k for the domain $\mu^{[N]*}(m_k) \leq 1$. We therefore define $\varphi_k^{(N)}(m) = \psi_k^\circ(\cdot)$ for $m \in G_k$, where ψ° denotes the upward convex envelope of ψ in the appropriate domain. We have

$$G_k = \{m : \mu^{[N]*}(m) \leq 1\}.$$

Suppose that $\varphi_{j+1}^{(i+1)}$ and G_{j+1} have already been constructed; we then have $i+1 > g$, $\tau_{j+1} < t^{[i+1]}$. We begin with the case $\tau_{j+1} > t^{[i]}$, defining $G_j = G_{j+1}$,

$$\Delta\psi_j(m) = I(\tau_j, \tau_{j+1}, m), \ \psi_j(m) = \Delta\psi_j(m) + \\ + \varphi_{j+1}^{(i+1)}(m), \ \varphi_j^{(i+1)}(m) = \psi_j^\circ(\cdot), \ m \in G_j$$

Consider the case $\tau_{j+1} = t^{[j]}$. Then $\psi_j(m^*) = \max[\Delta\psi_j(m^*) + \varphi_{j+1}^{(i+1)}(m_*)]$ for $m^* = m_* + m$, $m_* \in G_{j+1}$, $m = X^T(t^{[i]}, \vartheta) D^{[i]T} l$, $\mu^{[i]*}(l) \leq 1$.

The domain G_j is the set of all such vectors. Then $\varphi_j^{(i)}(m) = \psi_j^\circ(\cdot)$ for $m \in G_j$. The proof that this induction step is valid, as in the case $j = k$, is based on maximization of the appropriate conditional expectation of $\psi_j(\cdot)$. The construction continues until the time $\tau_1 = \tau_*$, when two cases may occur.

In the first case $\tau_* = t^{[g]}$. We then obtain a domain $G_1^{(g)}$ and a function $\varphi_1^{(g)}(m)$ for $m \in G_1^{(g)}$ which determines $\varkappa(\cdot)$ (2.7) so that

$$\varkappa(\tau_*, w_*, \Delta, \eta) = \varphi_1^{(g)}(m), \quad m \in G^{(g)} \tag{2.8}$$

In the second case $\tau_* = t^{[g]}$. We then obtain a domain $G_1^{(g+1)}$ and a function $\varphi_1^{(g+1)}(m)$ such that

$$\varkappa(\tau_*, w_*, \Delta, \eta) = \varphi_1^{(g+1)}(m), \quad m \in G^{(g+1)} \tag{2.9}$$

In the first case we have $m^* = m_* \in G_1^{(g)}$ in (2.6). In the second case $m^* = m_* + m$, where $m_* \in G_1^{(g+1)}$, $m = X^T(\tau_*, \vartheta) D^{[g]T} l$, $\mu^{[g]*}(l) \leq 1$.

3. CONSTRUCTION OF OPTIMAL STRATEGIES

To construct S_0^u as in (1.9) we have to define the functions

$$U_y^\circ(\cdot), \ U_y^*(\cdot), \ V_y^*(\cdot), \ p_y^\circ(\cdot), \ p_y^{*\circ}(\cdot)$$

and $q_y^{*\circ}(\cdot)$ of (1.3). The sets $U_y^\circ(\varepsilon)$, $V_y^{*\circ}(\varepsilon)$ and functions $p_y^\circ(t, x, y, \varepsilon)$, $q_y^{*\circ}(t, x, y, \varepsilon)$ are determined by certain conditions [7, p. 188] which guarantee that the motions (1.6) of the object and (1.7) of the pilot model will be close together. Indeed, the sets of numbers

$$p_r^\circ = p_r^\circ(\tau_*, x_*, y_*, \varepsilon) \quad \text{and} \quad q_s^{*\circ} = q_s^{*\circ}(\tau_*, x_*, y_*, \varepsilon)$$

are chosen subject to the conditions

$$\max_q \sum_{r,s=1}^{L_\eta, M_\eta} \langle (x_* - y_*) \cdot f(\tau_*, u^{[r]}, v^{[s]}) \, p_r^\circ q_s \rangle = \min_p \text{Idem}\,(p^\circ \to p) \tag{3.1}$$

$$\min_p \sum_{r,s=1}^{L_\eta, M_\eta} \langle (x_* - y_*) f(\tau_*, u^{[r]}, v^{[s]}) \, p_r q_s^{*\circ} \rangle = \max_q \text{Idem}\,(q^{*\circ} \to q) \tag{3.2}$$

on the assumption that relations (2.2) hold with $\eta = \eta(\varepsilon)$.

We define $U_y^\circ(\varepsilon) = U_y^{*\circ}(\varepsilon) = U_{\eta(\varepsilon)}$, $V_y^{*\circ}(\varepsilon) = V_{\eta(\varepsilon)}$, where $U_{\eta(\varepsilon)}$ and $V_{\eta(\varepsilon)}$ are the sets of vectors $\{u^{[r]}\}$ and $\{v^{[s]}\}$ are in (2.1), (3.1) and (3.2).

It remains to define the function

$$p_v^{*\circ}(\cdot) = \left\{p^{*\circ}(t, x, y, \varepsilon) = \left\{p_r^{*\circ} \geqslant 0, \quad \sum_{r=1}^{L_\eta} p_r^{*\circ} = 1\right\}\right\} \tag{3.3}$$

Relying on (2.4)–(2.9), we construct

$$\begin{aligned} \rho^\varepsilon(\tau_*, y_*) &= \langle m^{*\circ}(\tau_*, y_*, \varepsilon) X(\vartheta, \tau_*) y_*\rangle + \\ &+ \varkappa(\tau_*, m_*^\circ, \Delta, \eta(\varepsilon)) - \beta(\varepsilon, \tau_*)\,(1 + |X^T(\vartheta, \tau_*) m^{*\circ}(\tau_*, y_*, \varepsilon)|^2)^{1/2} = \\ &= \max_{m^*} \operatorname{Idem}(m^{*\circ}(\tau_*, y_*, \varepsilon) \to m^*, m_*^\circ \to m_*) \end{aligned} \tag{3.4}$$

$$\beta^2(\varepsilon, \tau_*) = (\varepsilon + \varepsilon(\tau_* - t_0))\exp\{2\lambda(\tau_* - t_0)\}$$

$$\lambda = \max_{t_* \leqslant t \leqslant \vartheta} |A(t)|, \quad |A(t)| = \max_{|x| \leqslant 1} |A(t)x|$$

The maximum in (3.4) is evaluated over the vectors m^* in the domain defined at the end of Sec. 2. Using the convexity of $\varkappa(\tau_*, m, \Delta, \eta)$ as a function of m, one can show that the vector $m^{*\circ}(\tau_*, y_*, \varepsilon)$ is uniquely defined by (3.4).

The optimal set $p_r^{*\circ}$ of (3.3) is defined by

$$\begin{aligned} \max_q \left\langle X^T(\vartheta, \tau_*) m^{*\circ}(\tau_*, y_*, \varepsilon) \sum_{r, s=1}^{L_\eta, M_\eta} f(\tau_*, u^{[r]}, v^{[s]})\, p_r^{*\circ} q_s\right\rangle = \\ = \min_p \operatorname{Idem}(p^{*\circ} \to p) \end{aligned} \tag{3.5}$$

This completes the construction of S_0^u.

The optimal strategy S_0^v of (1.4), (1.10) is constructed in a similar way; the necessary changes in (3.1), (3.2) are to replace y by z, interchange p and q and replace the minus sign in (3.4) by a plus. The optimal set $q_s^{*\circ} = q_s^{*\circ}(\tau_*, x_*, z_*, \varepsilon)$ is defined by

$$\min_p \left\langle X^T(\vartheta, \tau_*) m_v^{*\circ} \sum_{r, s=1}^{L_\eta, M_\eta} f(\tau_*, u^{[r]}, v^{[s]})\, p_r q_s^{*\circ}\right\rangle = \max_q \operatorname{Idem}(q^{*\circ} \to q) \tag{3.6}$$

where $m_v^{*\circ}$ is a maximizing vector. Unlike (3.4), the function $\rho_v^\circ(\tau_*, z_*)$ may not be convex, so the vector $m_v^{*\circ}$ is not unique. We must therefore take one of the maximizing vectors $m_v^{*\circ}$.

4. EXAMPLE

We will illustrate the computation of the payoff $\rho^\circ(t_*, x_*)$ (2.4) of the game by the following example: let the equation of motion of the object (1.1) be

$$d^2h/dt^2 = a(t)u + b(t)(u+v)^2 + c(t)v = F(t, u, v) \tag{4.1}$$

where h is a scalar, $a(t)$, $b(t)$ and $c(t)$ piecewise continuous functions $u \in W_u = \{u\colon u^{[1]} = -1,\ u^{[2]} = 1\}$, $v \in W_v = \{v\colon v^{[1]} = -1,\ v^{[2]} = 1\}$.

The quality criterion γ of (1.2) is

$$\gamma = |h[t^{[1]}]| + |h[\vartheta]|, \quad t^{[1]} \in [t_0, \vartheta) \tag{4.2}$$

Equation (4.1) may be reduced to normal form:

$$x^\cdot = Ax + f(t, u, v) \tag{4.3}$$

$$x = \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix}, \quad A = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}$$

$$f(t, u, v) = \begin{Vmatrix} 0 \\ F(t, u, v) \end{Vmatrix}$$

Then, by (1.2) we have

$$\gamma=|Dx[t^{[1]}]|+|Dx[t^{[2]}]| \tag{4.4}$$

$$D=\|1\ 0\|,\quad t^{[2]}=\vartheta.$$

We introduce the scalars $l^{(i)}$, $i=1, 2$, $|l^{(i)}|\leqslant 1$, $p_r=P(u=u^{[r]})$, $r=1, 2$, $q_s=P(v=v^{[s]})$, $s=1, 2$; $P(\cdots)$ denotes probability. Define the vectors

$$m^{(i)}=X^{\mathrm{T}}(\vartheta,\ t^{(i)})D^{\mathrm{T}}l^{(i)},\quad i=1,\ 2 \tag{4.5}$$

where

$$X(t,\tau)=\left\|\begin{matrix}1 & t-\tau\\ 0 & 1\end{matrix}\right\|-$$

is a fundamental matrix. Let $\{t_*, x_*\}$, $t_*<t^{[1]}$ be the position of the object (4.3). We will determined $\rho^\circ(t_*, x_*)$. Define

$$t_0=0,\quad \vartheta=4,\quad t^{(1)}=1,\quad b(t)\equiv{}^1/_2$$

$$a(t)=\begin{cases}4, & t_0\leqslant t<2\\ 0, & 2\leqslant t\leqslant\vartheta\end{cases},\quad c(t)=\begin{cases}0, & t_0\leqslant t<3\\ 2, & 3\leqslant t\leqslant\vartheta\end{cases}$$

Following the construction of Sec. 2, compute the function

$$\Delta\psi_k(m^{(2)})=l(\tau_k,\tau_{k+1},m^{(2)})=(\tau_{k+1}-\tau_k)\Big(\min_p\max_q m^{(2)\mathrm{T}}\times$$

$$\times\sum_{r,s=1}^{2}X(\vartheta,\tau_k)f(\tau_k,u^{[r]},v^{[s]})p_rq_s\Big)=(\tau_{k+1}-\tau_k)\,|l^{(2)}|\,\vartheta-\tau_k)\,\mathrm{extr}(\tau_k) \tag{4.6}$$

$$\mathrm{extr}(\tau_k)=\min_p\max_q\sum_{r,s=1}^{2}F(\tau_k,u^{[r]},v^{[s]})p_rq_s=2$$

for $p_1^\circ=1$, $p_2^\circ=0$, $q_1^\circ=0$, $q_2^\circ=1$. Then $\varphi_k(m^{(2)})=2(\tau_{k+1}-\tau_k)\ (\vartheta-\tau_k)$. Next, letting $\tau\in[3, 4]$, we have $\mathrm{extr}(\tau_j)=2$, $p_1^\circ=1$, $p_2^\circ=0$, $q_1^\circ=0$, $q_2^\circ=1$. For $\tau_j\in[2, 3]$ we have $\mathrm{extr}(\tau_j)=1$, $p_1^\circ=p_2^\circ=q_1^\circ=q_2^\circ=½$. Therefore, by induction on j, proceeding from $j=k$ to $j=d$, where τ_d is the least of the times $\tau_j\geqslant 2$, we obtain

$$\varphi_d^{(2)}(m^{(2)})=\sum_{j=d}^{k}(\tau_{j+1}-\tau_j)\,(\vartheta-\tau_j)\,\mathrm{extr}(\tau_j) \tag{4.7}$$

If $\tau_j\in[1, 2]$, we have $\mathrm{extr}(\tau_j)=-2$, $p_1^\circ=1$, $p_2^\circ=0$, $q_1^\circ=1$, $q_2^\circ=0$. Therefore, proceeding by induction on j from $j=d$ to $j=g$, where τ_g is the best of the times $\tau_j\geqslant t^{[1]}=1$, we obtain

$$\varphi_g^{(2)}(m^{(2)})=\varphi_d^{(2)}(m^{(2)})+\sum_{j=g}^{d-1}(\tau_{j+1}-\tau_j)\,|l^{(2)}|\,(\vartheta-\tau_j)\,(-2) \tag{4.8}$$

If $\tau_j<t^{[1]}$, then $\mathrm{extr}(\tau_j)=-2$, $p_1^\circ=q_1^\circ=1$, $p_2^\circ=q_2^\circ=0$. Proceeding by induction on j from $j=g$ to $j=1$, we see that the function $\varkappa(\cdot)$ of (2.6) is defined by

$$\varkappa(t_*,m^{(1)}+m^{(2)})=\varphi_1(m^{(1)}+m^{(2)})=\sum_{j=1}^{g-1}(\tau_{j+1}-\tau_j)\,|l^{(1)}(t^{[1]}-\tau_j)+ \tag{4.9}$$

$$+l^{(2)}(\vartheta-\tau_j)\,|\,\mathrm{extr}(\tau_j)+\varphi_g^{(2)}(m^{(2)})$$

and finally

$$\rho^\circ(t^*,x_*)=\max_{m^{(1)}+m^{(2)}}\left[(m^{(1)}+m^{(2)})^{\mathrm{T}}X(\vartheta,t_*)x_*+\varkappa(t_*,m^{(1)}+m^{(2)})\right],\quad (m^{(1)}+m^{(2)})\in G$$

$$G=\left\{m^{(1)}+m^{(2)}=\left\|\begin{matrix}l^{(1)}\\ (t^{[1]}-\vartheta)l^{(1)}\end{matrix}\right\|+\left\|\begin{matrix}l^{(2)}\\ 0\end{matrix}\right\|:\ |l^{(i)}|\leqslant 1,\ i=1,2\right\} \tag{4.10}$$

If $\{t_*, x_*\}$, $t_*>t^{[1]}$, we have

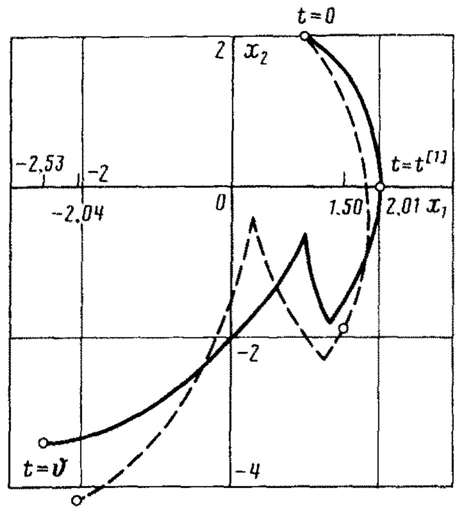

FIG. 1.

$$\rho^\circ(t_*, x_*) = \max_{m^{(2)}} [m^{(2)\tau} X(\vartheta, t_*) x_* + \varkappa(t_*, m^{(2)})] \tag{4.11}$$

$$\varkappa(t_*, m^{(2)}) = \begin{cases} \varphi_g^{(2)}(m^{(2)}), & 1 < t_* < 2 \\ \varphi_d^{(2)}(m^{(2)}), & 2 \leqslant t_* < 3 \\ \sum_{j=e}^{k} 2(\tau_{j+1} - \tau_j)(\vartheta - \tau_j), & 3 \leqslant t_* \leqslant 4 \end{cases}$$

where τ_e is the least of the times $\tau_j \geqslant 3$.

If $\{t_*, x_*\}$, $t_* = t^{[1]}$, we have

$$\rho^\circ(t_*, x_*) = \max_{m^{(1)}+m^{(2)}} [(m^{(1)} + m^{(2)})^\tau X(\vartheta, t_*) x_* + \\ + \varkappa(t_*, m^{(2)})], \quad \varkappa(t_*, m^{(2)}) = \varphi_g^{(2)}(m^{(2)}) \tag{4.12}$$

In this example, if we let $\delta \to 0$ we get explicit expressions for $\varkappa(\cdot)$ in closed form, as integrals. Here we have preferred the discrete-sum representation of these functions, so as to illustrate the general features of the method through a simple, specific model.

Once the payoff $\rho^\circ(t, x)$ has been determined as in (4.10)–(4.12), the optimal strategies $\{S_0^u, S_0^v\}$ can be constructed as described in Sec. 3.

The control process for the object (4.3) with quality index (4.4) was simulated on a computer for initial data $t_* = 0$, $x_* = \{1, 2\}$. The solid curve in Fig. 1 represents the motion of the object (4.3) with $S_0^u = \{u^{[r]}, p_{yr}^\circ\}$, $S_0^v = \{v^{[s]}, q_{zs}^\circ\}$ (the quality index γ of (4.4) is almost identical with the payoff of the game $\rho^\circ(t_*, x_*) = 4.5$); the dashed curve represents the motion of the object with S_0^u, $S^v = \{v^{[s]}, q_{zs} = 1/2\} \neq S_0^v$ [$\gamma = 3.5 < \rho^\circ(t_*, x_*)$].

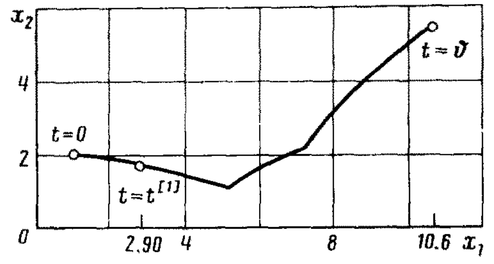

FIG. 2.

Figure 2 shows the motion of the object for $S^u = \{u^{[r]}, p_{y1} = 0.7, p_{y2} = 0.3\} \neq S_0^u$, S_0^v $[\gamma = 13.5 > \rho^\circ(t_*, x_*)]$.

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THREE-DIMENSIONAL MOTION OF A MATERIAL POINT†

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The two-dimensional motion of a material point over the active portion of its trajectory can be generalized in a natural way to three dimensions. Corresponding to the traditional flight plane, to which the trajectory of motion is confined in three dimensions, we have a set of flight surfaces obtained from it by bending. The three-dimensional system of differential equations governing the motion of a material point splits into a two-dimensional system, which describes the motion in the flight surface, and a system of ordinary differential equations, which describes the bending of the surface. By solving this system of equations one can determine by analytical means how the velocity and coordinate vectors over the active portion of the trajectory depend on its three-dimensional distortion. The results obtained may be used to analyse the three-dimensional motion of a material point, to select trajectories in space and to control the three-dimensional motion of the centre of mass over the active portions. In some cases one can actually derive analytical expressions for solutions to boundary-value and extremal problems associated with the three-dimensional motion of a material point.

1. THE BASIS TRIHEDRON AND THE DIFFERENTIAL EQUATIONS OF ITS ROTATION

WE SHALL be concerned with the three-dimensional motion of a material point over the active portion of its trajectory about a single attractive centre. A physical example of such a motion is that

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